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An exact solution of the Navier–Stokes equation which describes non-orthogonal stagnation-point flow in two dimensions

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A similarity solution is found which describes the flow impinging on a flat wall at an arbitrary angle of incidence. The technique is similar to a method used by Jeffery (1915) and discussed more recently by Peregrine (1981).

1. Introduction

Exact solutions of the Navier–Stokes equation are exceptionally rare in fluid mechanics because of the analytic difficulties associated with nonlinear boundary-value problems. One of the primary difficulties rests in the fact that nonlinear problems do not admit a superposition principle, thereby ruling out the building up of complicated solutions from simple ones. This severely restricts our scope to problems having particularly simple geometries for which a similarity solution exists.

One of the few flows which admits a similarity solution is two-dimensional stagnation-point flow. The solution was first proposed by Blasius (1908) and the resulting differential equation was integrated by Hiemenz (1911). The numerical work was later improved by Howarth (1935). The solution is sketched here for future reference.

We assume that the fluid is incompressible and the flow steady. The governing equations are given by

$$\nabla \cdot \boldsymbol{q} = 0, \tag{1.1}$$

$$(\boldsymbol{q} \cdot \nabla)\, \boldsymbol{q} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \boldsymbol{q}, \tag{1.2}$$

where $\boldsymbol{q}(x,y)$, $p(x,y)$, ρ, ν are respectively fluid velocity, pressure, density and viscosity, the latter two being constants.

Equation (1.1) is satisfied identically by the introduction of a stream function $\psi(x,y)$ defined by

$$\boldsymbol{q}(x,y) = \nabla \times [\psi(x,y)\, \hat{k}]. \tag{1.3}$$

After substituting (1.3) into (1.2) and then taking the curl to eliminate the pressure term, we obtain the vorticity-transport equation

$$\nu \nabla^4 \psi = \frac{\partial \psi}{\partial y}\frac{\partial \nabla^2 \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \nabla^2 \psi}{\partial y}. \tag{1.4}$$

In stagnation-point flow a rigid wall occupies the entire x-axis and the fluid domain is $y > 0$. The boundary conditions on $\boldsymbol{q}(x,y)$ are

$$\left.\begin{aligned} &\boldsymbol{q}(x,0) = \boldsymbol{0}, \\ &\boldsymbol{q}(x,y) \sim \beta x \hat{\imath} - \beta y \hat{\jmath}, \quad \text{as } y \to \infty, \end{aligned}\right\} \tag{1.5}$$

where β has units of inverse time. If the variables (x, y) and ψ are non-dimensionalized using $(\nu/\beta)^{\frac{1}{2}}$ and ν respectively, the scaled vorticity-transport equation is identical with (1.4) except that the viscosity coefficient is missing from the left side of the equation. The non-dimensional boundary conditions are given by

$$\left.\begin{aligned} \psi(x,0) &= \frac{\partial\psi}{\partial y}(x,0) = 0, \\ \psi(x,y) &\sim xy, \quad \text{as } y \to \infty. \end{aligned}\right\} \tag{1.6}$$

Blasius showed that the solution to this problem was of the form

$$\psi(x,y) = xF(y). \tag{1.7}$$

The boundary-value problem for $F(y)$ obtained by substituting (1.7) into the governing equation is, after one integration,

$$\left.\begin{aligned} &F'''(y) + F(y)\,F''(y) - F'(y)^2 = -1, \\ &F(0) = F'(0) = 0, \\ &F'(\infty) = 1. \end{aligned}\right\} \tag{1.8}$$

From a numerical solution recorded by Goldstein (1965), we observe that, for small y,

$$F(y) = \tfrac{1}{2}Cy^2 - \tfrac{1}{6}y^3 + O(y^5), \tag{1.9}$$

where $C = 1.232588$.

The asymptotic behaviour for large y is given by

$$F(y) \sim y - A + O\{(y-A)^{-4}\exp[-\tfrac{1}{2}(y-A)^2]\}, \tag{1.10}$$

where $A = 0.647900$.

2. Non-orthogonal stagnation-point flow

A number of exact solutions to the Navier–Stokes equation are known for which the nonlinear convective terms vanish identically. One of these is linear shear flow along a flat wall whose stream function is given by

$$\psi(x,y) = \tfrac{1}{2}y^2. \tag{2.1}$$

In a very real sense this flow is a kind of orthogonal counterpart to the flow mentioned in §1. Whereas stagnation-point flow describes the motion of a fluid against a flat wall, shear flow describes the motion of the fluid along the wall. This leads to an interesting question: is it possible to combine these two flows in a way which yields a flow impinging on the wall at some angle of incidence? The nonlinearity of the governing equations clearly forbids a linear combination of the two solutions. However a linear combination of the far-field behaviours of the two flows satisfies the Navier–Stokes equation exactly as shown below.

Consider the flow described by

$$\begin{aligned} \psi(x,y) &= \tfrac{1}{2}y^2\cos\alpha + xy\sin\alpha \\ &= \tfrac{1}{2}y\cos\alpha(y + 2x\tan\alpha), \end{aligned} \tag{2.2}$$

where $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$ is a parameter. It is trivial to show that this stream function satisfies the vorticity-transport equation (1.4) exactly. A sketch of the streamlines

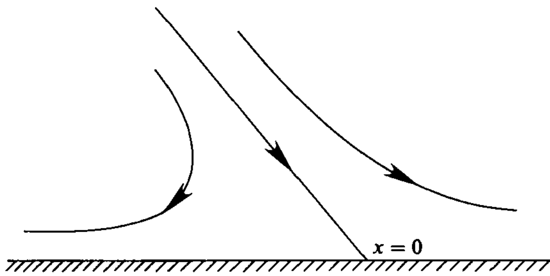

FIGURE 1. Streamlines for the flow to which non-orthogonal stagnation-point flow asymptotes. The dividing streamline $\psi = 0$ is a straight line with slope $m = -2 \tan\alpha$.

as depicted in figure 1 reveals a kind of non-orthogonal stagnation-point flow in which the slope of the dividing streamline is

$$m = -2 \tan\alpha. \tag{2.3}$$

By regarding the stream function in (2.2) as the far-field behaviour of a flow impinging obliquely on a flat wall, we are led to the following similarity solution for such a flow:

$$\psi(x, y) = g(y) + xf(y). \tag{2.4}$$

We shall refer to $f(y)$ as the normal component of the flow and $g(y)$ as the tangential component. Both components must satisfy no-slip conditions at the wall and conditions consistent with (2.2) at infinity.

3. Normal component

The substitution of (2.4) into the vorticity-transport equation yields fourth-order differential equations for the two functions. The equation for $f(y)$ is identical with that derived by Blasius and one integration of it yields

$$\left.\begin{aligned} &f'''(y) + f(y)f''(y) - f'(y)^2 + \sin^2\alpha = 0,\\ &f(0) = f'(0) = 0,\\ &f'(\infty) = \sin\alpha. \end{aligned}\right\} \tag{3.1}$$

The solution of this equation is homologous to that of Hiemenz. This is shown by assuming a solution of the form

$$f(y) = aF(ay) \tag{3.2}$$

where a is a constant of homology and $F(y)$ is defined by (1.8). The substitution of (3.2) into (3.1) reveals that

$$a = (\sin\alpha)^{\frac{1}{2}}. \tag{3.3}$$

A check of the boundary conditions yields the following results:

$$\left.\begin{aligned} &f(0) = aF(0) = 0,\\ &f'(0) = a^2F'(0) = 0,\\ &f'(\infty) = a^2F'(\infty) = a^2 = \sin\alpha. \end{aligned}\right\} \tag{3.4}$$

Thus $f(y)$ as defined in (3.2) and (3.3) satisfies (3.1). It follows that, for small y,

$$f(y) = \tfrac{1}{2}C(\sin\alpha)^{\frac{3}{2}}\, y^2 - \tfrac{1}{6}\sin^2\alpha\; y^3 + O(y^5). \tag{3.5}$$

The asymptotic behaviour of $f(y)$ for large y is given by

$$f(y) \sim y \sin\alpha - A(\sin\alpha)^{\frac{1}{2}} + \text{exponentially small terms.} \tag{3.6}$$

When the parameter $\alpha = \frac{1}{2}\pi$, we have $f(y) = F(y)$ and orthogonal stagnation-point flow is recovered. In the limiting case $\alpha = 0$, the component $f(y)$ vanishes identically. The case $\alpha = \frac{1}{6}\pi$ occurs in the problem of flow through a linearly constricted channel as considered by Smith (1976).

4. Tangential component

The corresponding boundary-value problem for the tangential component $g(y)$ is linear and is given by

$$\left.\begin{aligned} &g^{\text{iv}}(y) + f(y)\,g'''(y) - f''(y)\,g'(y) = 0,\\ &g(0) = g'(0) = 0,\\ &g''(\infty) = \cos\alpha. \end{aligned}\right\} \tag{4.1}$$

As in the previous case this equation can be integrated once. The asymptotic behaviours of $f(y)$ and $g(y)$ for large y are used to determine the constant of integration. The resulting third-order equation is

$$g'''(y) + f(y)\,g''(y) - f'(y)\,g'(y) = -A(\sin\alpha)^{\frac{1}{2}}\cos\alpha, \tag{4.2}$$

When the parameter $\alpha = \frac{1}{2}\pi$, both the equation and its boundary conditions become homogeneous. The solution therefore is the trivial one. On the other hand the limiting case $\alpha = 0$ sees all terms in (4.2) vanishing except for the third derivative. The solution which corresponds to the boundary conditions in (4.1) is the original linear shear flow given in (2.1).

The solution for general α is accomplished using two transformations. In the first we let

$$g'(y) = h(y)\cos\alpha, \tag{4.3}$$

and thereby reduce the order of the equation. The problem for $h(y)$ is given by

$$\left.\begin{aligned} &h''(y) + f(y)\,h'(y) - f'(y)\,h(y) = -A(\sin\alpha)^{\frac{1}{2}},\\ &h(0) = 0,\\ &h'(\infty) = 1. \end{aligned}\right\} \tag{4.4}$$

Equation (4.4) can be cleansed of its dependence on α using the substitutions

$$\left.\begin{aligned} &f(y) = aF(ay),\\ &h(y) = a^{-1}H(ay), \quad a \neq 0, \end{aligned}\right\} \tag{4.5}$$

where $a = (\sin\alpha)^{\frac{1}{2}}$ and $F(y)$ is Hiemenz's function. Introducing a new independent variable $z = ay$, we have

$$\left.\begin{aligned} &H''(z) + F(z)\,H'(z) - F'(z)\,H(z) = -A,\\ &H(0) = 0,\\ &H'(\infty) = 1. \end{aligned}\right\} \tag{4.6}$$

Two observations can be made which enable us to solve this equation exactly. First, the function $AF'(z)$ is a particular solution of the equation and, secondly, $F''(z)$ is a solution of the corresponding homogeneous problem. Reduction of order is used to generate the second fundamental solution of the homogeneous equation and in this way the general solution of (4.6) is pieced together. Unfortunately substantial difficulties are encountered in satisfying the boundary condition at infinity and it is much easier to integrate (4.6) numerically using a shooting method to satisfy the far-field condition. This yields the result

$$H'(0) = 1.406544 = D. \tag{4.7}$$

We can now use (4.7) in place of the condition at infinity and solve (4.6) as an initial-value problem. The closed-form solution for $H(z)$ so obtained is

$$H(z) = AF'(z) + C(D-AC)\,F''(z)\int_0^z w(t)\,\mathrm{d}t, \tag{4.8}$$

where
$$w(t) = F''(t)^{-2}\exp\left\{-\int_0^t F(s)\,\mathrm{d}s\right\}.$$

For large z, the behaviour of $H(z)$ is given by

$$H(z) \sim z + O\{(z-A)^{-2}\ln(z-A)\exp[-\tfrac{1}{2}(z-A)^2]\}. \tag{4.9}$$

It follows from (4.3) and (4.5) that the solution of (4.2) is given by

$$g(y) = a^{-1}\cos\alpha\int_0^y H(ay)\,\mathrm{d}y. \tag{4.10}$$

When y is small, we have

$$g(y) = \tfrac{1}{2}D\cos\alpha\, y^2 - \tfrac{1}{6}A(\sin\alpha)^{\frac{1}{2}}\cos\alpha\, y^3 + O(y^5). \tag{4.11}$$

When $\alpha \neq 0$, the asymptotic behaviour of $g(y)$ for large y is given by

$$g(y) \sim \tfrac{1}{2}y^2\cos\alpha + B\cot\alpha + \text{exponentially small terms}, \tag{4.12}$$

where $B = 0.215395$.

It was noted earlier that, in the case $\alpha = 0$, we have $g(y) = \frac{1}{2}y^2$. It can be shown, however, that, as $\alpha \to 0$, all terms in (4.11), except for the first, vanish with the result

$$\lim_{\alpha\to 0} g(y) = \tfrac{1}{2}Dy^2. \tag{4.13}$$

This paradox can be explained by examining the structure of the flow when α is small.

Equation (4.6) consists of a viscous term, two convection terms and a forcing term. The forcing constant can be identified with the convection terms. When α is small, the solution to (4.6) consists of three regions: an inner region where $z \simeq \alpha^{\frac{1}{2}}y \ll 1$, a transition layer where $\alpha^{\frac{1}{2}}y = O(1)$, and an outer region where $\alpha^{\frac{1}{2}}y \gg 1$. In both the inner and outer regions, the viscous term $H''(z)$ dominates the inertial terms and a shear flow is obtained: $H(z) \simeq Dz$ in the inner region, $H(z) \simeq z$ in the outer region. In the transition layer inertial effects are significant and the shear rate is gradually changed from its value of D near the wall to its value of 1 far from the wall. As $\alpha \to 0$, both the transition layer which is located at $y = O(\alpha^{-\frac{1}{2}})$ and the outer region are pushed off to infinity leaving simply the shear flow in the inner region which now extends from the wall to $+\infty$. Thus the result in (4.13) is obtained. On the other hand when $\alpha = 0$, the convection terms in (4.4) vanish identically and a transition layer cannot exist. The outer shear flow is the only one which can satisfy both the boundary conditions at the wall and at infinity. The solution $g(y) = \frac{1}{2}y^2$ corresponding to $\alpha = 0$ is therefore a singular limit.

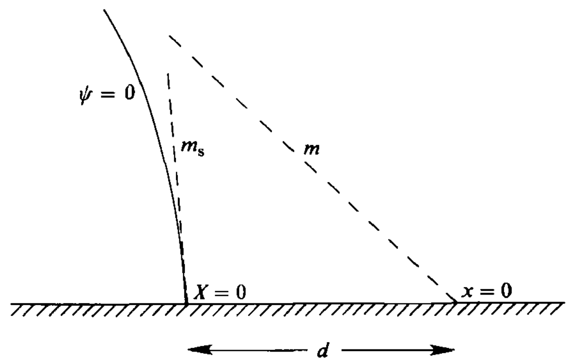

FIGURE 2. The dividing streamline $\psi = 0$ meets the wall at $X = 0$ and has slope m_s at that point. The ratio of m_s to m is the same for all angles of incidence.

5. Behaviour of the flow near the wall

The flow near the wall can be analysed by substituting the small-y expansions for $f(y)$ and $g(y)$ into (2.4). If we define a new horizontal coordinate X in the following way

$$X = x + \frac{D}{C} \cos\alpha \, (\sin\alpha)^{-\frac{3}{2}}, \tag{5.1}$$

the resulting expansion for the stream function is given by

$$\psi(X, y) = \frac{1}{6}\left(\frac{D}{C} - A\right)(\sin\alpha)^{\frac{1}{2}} \cos\alpha \, y^2 \left\{ y + 3C \tan\alpha \left(\frac{D}{C} - A\right)^{-1} X - (\sin\alpha)^{\frac{1}{2}} \tan\alpha \left(\frac{D}{C} - A\right)^{-1} Xy + O(y^3) \right\}. \tag{5.2}$$

Recall from §2 that far from the wall the dividing streamline $\psi = 0$ is a straight line with slope $m = -2\tan\alpha$ which if extended would intersect the wall at $x = 0$. From (5.2) we see that in fact the streamline $\psi = 0$ meets the wall at $X = 0$ (see figure 2). The distance between these two locations is, from (5.1),

$$d = 1.141\,131\,(1 + k^2)^{\frac{1}{4}} k^{-\frac{3}{2}}, \tag{5.3}$$

where $k = \frac{1}{2}|m|$. As $|m| \to \infty$, orthogonal stagnation-point flow is approached and we have $d \to 0$. On the other hand as $m \to 0$, the impinging stream comes into the wall more and more obliquely and $d \to \infty$. If we regard the flow to the left of the dividing streamline as a region of separated flow, we see that the effect of an almost tangential impinging stream is to push this region of separated flow off to $x = -\infty$. The flow in the vicinity of $x = 0$ then essentially becomes a shear flow parallel to the wall.

A second truly remarkable result is the value of the slope m_s of the dividing streamline at the point $X = 0$. From (5.2) we see that

$$m_s = \frac{-3C^2}{D - AC} \tan\alpha. \tag{5.4}$$

Thus the ratio m_s/m is found to be

$$\frac{m_s}{m} = \frac{1.5C^2}{D - AC} = 3.748513, \tag{5.5}$$

which is independent of α! The slope of the dividing streamline $\psi = 0$ at the wall divided by its slope at infinity is the same for all non-orthogonal stagnation-point flows. That these flows are all exact solutions of the Navier–Stokes equation suggests that this constant is somehow intrinsic to the governing equation of fluid mechanics.

The referees are thanked for their suggestions which led to the solution of (4.6).

I would like to dedicate this paper to the memory of my wife Margaret who passed away 3 June 1985.

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Note added in proof by the author

As this paper went to press, it was discovered that the problem it concerns has already been discussed by Stuart (1959) and Tamada (1979). The results in the three papers are consistent. The present paper, unlike the previous two, formulates the problem so that the normal and tangential components, $f(y)$ and $g(y)$, depend on the angle at which the incident flow impinges on the wall $y = 0$. As a consequence equation (4.2) for $g(y)$ is non-homogeneous while the corresponding equations of Stuart and Tamada are homogeneous. That the two formulations are equivalent is brought out by the agreement between the solution provided by Stuart in equation (11) of his paper and the homogeneous part of my solution as recorded in (4.8). The main contributions of the present paper are contained in §5. These include the point of attachment of the dividing streamline as recorded in (5.3) and the existence of a universal constant given in (5.5) for this class of flows.

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